

MTA

NOTATIONS :

\mathbb{Z} = Set of integers.

\mathbb{N} = Set of natural numbers = $\{n \in \mathbb{Z} : n \geq 1\}$.

\mathbb{Q} = Set of rationals.

\mathbb{R} = Set of real numbers.

\mathbb{C} = Set of complex numbers.

1. Prove that the following limit exists and find the limit:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right).$$

2. Suppose $f: [a, b] \rightarrow \mathbb{R}$ is continuous. Then for any $r_1, r_2, \dots, r_n \in f[a, b]$, prove that there exists $x \in [a, b]$ such that $f(x) = \frac{r_1 + r_2 + \cdots + r_n}{n}$.
3. Suppose $y(x) = x^2$ is a solution of $y'' + P(x)y' + Q(x)y = 0$ on $(0, 1)$ where P and Q are continuous functions on $(0, 1)$. Can both P and Q be bounded functions. Justify your answer.
4. For each $\alpha > 0$, find all pairs of $(x_0, y_0) \in \mathbb{R}^2$ such that the following initial value problem has a unique solution in the neighbourhood of (x_0, y_0)

$$y' = y^\alpha ; y(x_0) = y_0.$$

5. Let $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\| \leq 1\}$, and let

$$f(x) = \inf\{\|x - y\| : y \in B\}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

If $F(x) = \max\{1 - f(x), 0\}$, $x = (x_1, x_2) \in \mathbb{R}^2$, then prove that

$$\lim_{n \rightarrow \infty} \int \int_{\mathbb{R}^2} F^n(x_1, x_2) dx_1 dx_2 = \pi.$$

6. Let $f : X \rightarrow Y$ be a function from a metric space (X, d_1) to a compact metric space (Y, d_2) . Let $G_f := \{(x, y) : y = f(x)\} \subset X \times Y$ denote the graph of f . Show that f is continuous iff G_f is closed in $X \times Y$. The metric d on $X \times Y$ is the product metric which is defined as $d((x_1, y_1), (x_2, y_2)) := \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}$.

7. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a Borel measurable function. Show that

$$\sum_{n=1}^{\infty} m(\{f \geq n\}) \leq \int f dm \leq \sum_{n=1}^{\infty} m(\{f > n\}),$$

where m denotes the Lebesgue measure.

8. Let (a_n) be a sequence of real numbers and m be the Lebesgue measure. Suppose $\frac{1}{n} \sum_{k=1}^n f(a_k) \rightarrow \int_{\mathbb{R}} f dm$ for all Lebesgue integrable functions f on \mathbb{R} . Prove that (a_n) is dense in \mathbb{R} .

9. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function i.e., analytic everywhere in \mathbb{C} . Suppose

$$\lim_{|z| \rightarrow \infty} \frac{f(z)}{z} = 0.$$

Prove that f is a constant function.

10. What are the holomorphic functions f on an open connected subset $\Omega \subset \mathbb{C}$ such that $g : \Omega \rightarrow \mathbb{C}$ defined by $g(z) = \operatorname{Re}(z)f(z)$ is also holomorphic.